Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 4, pp. 575–579, 2004

AVERAGING OF A LAYERED ELASTIC MEDIUM WITH LOW DYNAMIC DISSIPATION AT THE INTERLAYER BOUNDARY

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UDC 539.3

Averaged relations for a layered medium with dynamic dissipation at the interlayer boundary are constructed for dynamic problems of longitudinal shear and two-dimensional theory of elasticity. For low dissipation, the averaged boundary-value problem is shown to be singularly perturbed. The degeneration of the boundary-value problem is studied qualitatively.

Key words: *elasticity theory, longitudinal shear, averaging.*

In [1], we constructed averaged relations for an elastic medium with nonstandard matching conditions specified on the interlayer boundary, which relate, for example the shear stress to the shear-displacement discontinuity by a certain nonnegative coefficient called the friction factor. For a small friction factor, averaged boundary-value problems were shown to be singularly perturbed and some formulations of limiting problems were studied. In the present paper, unlike in [1], we study the case where the matching condition relating the shear stress to the shear-velocity discontinuity is specified on the interlayer boundary. It is clear that this formulation is meaningful only in dynamic elasticity problems. The matching condition for composite materials was first formulated in [2]. From a mechanical viewpoint, this condition models, in some sense, wave propagation in a medium in the presence of internal dissipation at the interlayer boundary. It was found that this matching condition changes the situation drastically. Unlike in the case studied in [1], where the averaged relations corresponded to a homogeneous anisotropic material (see also [3]), the averaging considered in the present paper leads to a viscoelastic material. As in [1], for a small friction factor, singular degeneration of the boundary-value problem occurs and the viscoelastic term vanishes in the limiting relations. A longitudinal shear problem and a two-dimensional elasticity problem are studied.

1. Longitudinal Shear. Bakhvalov and Panasenko [4] studied the problem of small longitudinal vibrations of a rod with allowance for dissipation. In the this section, we consider the averaging problem for a layered anisotropic elastic medium under conditions of longitudinal shear with dissipation at the interlayer boundary. The vibration equation is written as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_1}\Big(a_{11}\,\frac{\partial u}{\partial x_1} + a_{12}\,\frac{\partial u}{\partial x_2}\Big) - \frac{\partial}{\partial x_2}\Big(a_{12}\,\frac{\partial u}{\partial x_1} + a_{22}\,\frac{\partial u}{\partial x_2}\Big) = f. \tag{1.1}$$

We denote the maximum diameter of the domain Ω on the plane (in the x_1 direction) by L and set $\varepsilon = L/N$, where $N \gg 1$. It is assumed that Eq. (1.1) is uniformly hyperbolic, the layers are orthogonal to the x_1 axis, $x = (x_1, x_2)$, and the periodicity cell consists of two materials and coincides with the interval $(0, 1)$:

$$a_{ij}(\eta) = a^1_{ij}, \quad \eta \in (0, h), \qquad a_{ij}(\eta) = a^2_{ij}, \quad \eta \in (h, 1), \qquad i, j = 1, 2.$$

The coefficients $a_{ij} = a_{ij}(x_1/\varepsilon)$ ($i, j = 1$, 2) are measurable bounded periodic functions of the fast variable $\eta = x_1/\varepsilon$ with a period of 1 and there exists a constant $\alpha > 0$ such that

$$\sum_{s,l=1}^{2} a_{sl}\xi_s\xi_l \ge \alpha(\xi_1^2 + \xi_2^2)$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 4, pp. 140–146, July–August, 2004. Original article submitted October 24, 2003.

0021-8944/04/4504-0575

for any real ξ_i $(i = 1, 2)$. At the interface between two materials inside the cell, the following matching condition relating the normal stress and the rate of the process is satisfied:

$$a_{11}\,\frac{\partial u}{\partial x_1} + a_{12}\,\frac{\partial u}{\partial x_2}\Big|_{\eta=h\pm 0} = \frac{k}{\varepsilon}\Big[\frac{\partial u}{\partial t}\Big](h). \tag{1.2}$$

Here the brackets denote the discontinuity of the function at the interface between the materials:

$$\Big[\frac{\partial u}{\partial t}\Big] = \frac{\partial u}{\partial t}(h+0) - \frac{\partial u}{\partial t}(h-0).$$

For Eq. (1.1) subject to the matching conditions (1.2), we formulate the initial boundary-value problem in the domain $Q = (0,T) \times \Omega$:

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(x,t)\Big|_{\partial\Omega} = 0 \tag{1.3}$$

($\partial\Omega$ is the boundary of the domain Ω). The solution of problem (1.1)–(1.3) will be denoted by $u^\varepsilon(x,t)$. To construct the formal asymptotic solution of this problem, we use the averaging method proposed by Bakhvalov [4]. After introduction of the fast variable η, Eq. (1.1) becomes

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \Big(\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)\Big[a_{11}\Big(\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)u^\varepsilon + a_{12}\,\frac{\partial u^\varepsilon}{\partial x_2}\Big] - \frac{\partial}{\partial x_2}\Big[a_{12}\Big(\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}\Big)u^\varepsilon + a_{22}\,\frac{\partial u^\varepsilon}{\partial x_2}\Big] = f. \tag{1.4}$$

We seek a solution of the boundary-value problem in the form of a series in powers of ε:

$$u^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u^n(x,\eta,t). \tag{1.5}$$

To construct the averaged equation, it is necessary to determine only The first three terms of this series. Substituting (1.5) into (1.4), collecting terms of equal powers of ε, and performing transformations, we obtain the following equation for a power of ε^{-2}:

$$-\frac{\partial}{\partial \eta}\Big[a_{11}\,\frac{\partial u^0}{\partial \eta}\Big] = 0,$$

which implies that $u^0 = u^0(x,t)$. For a power of ε^{-1}, we have the following equation and the matching condition

$$-\frac{\partial}{\partial \eta}\Big[a_{11}\,\frac{\partial u^1}{\partial \eta} + a_{11}\,\frac{\partial u^0}{\partial x_1} + a_{12}\,\frac{\partial u^0}{\partial x_2}\Big] - \frac{\partial}{\partial x_2}\Big[a_{12}\,\frac{\partial u^0}{\partial \eta}\Big] = 0; \tag{1.6}$$

$$a_{11}\,\frac{\partial u^1}{\partial \eta} + a_{11}\,\frac{\partial u^0}{\partial x_1} + a_{12}\,\frac{\partial u^0}{\partial x_2}\Big|_{\eta=\pm 0} = k\Big[\frac{\partial u^1}{\partial t}\Big]. \tag{1.7}$$

From relations (1.6) and (1.7) it follows that the function $u^1(x,\eta,t)$ is uniquely determined (with accuracy up to an arbitrary constant). Relation (1.5) implies that

$$a_{11}\,\frac{\partial u^1}{\partial \eta} + a_{11}\,\frac{\partial u^0}{\partial x_1} + a_{12}\,\frac{\partial u^0}{\partial x_2} = \varphi(x,t).$$

Let us determine the function $\varphi(x,t)$. We set

$$\lambda_{11} = \int_0^1 \frac{ds}{a_{11}(s)}, \qquad \lambda_{12} = \int_0^1 \frac{a_{12}(s)}{a_{11}(s)}\,ds, \qquad \mu = \int_0^1 \Big[a_{22}(s) - \frac{a_{12}^2(s)}{a_{11}(s)}\Big]\,ds$$

and introduce the function

$$S(x,t) = \frac{\partial u^0}{\partial x_1} + \lambda_{12}\,\frac{\partial u^0}{\partial x_2}.$$

Using (1.6) and the matching condition (1.2), we obtain the following ordinary differential equation for the function $\varphi(x,t)$:

$$\varphi(x,t) + k\lambda_{11}\,\frac{\partial \varphi}{\partial t} = k\,\frac{\partial S}{\partial t}. \tag{1.8}$$

Introducing the new unknown function

$$\psi(x,t) = \lambda_{11}\varphi(x,t) - S(x,t),$$

we reduce Eq. (1.8) to the form

$$k\,\frac{\partial\psi}{\partial t}+\frac{\psi(x,t)+S}{\lambda_{11}}=0.$$

Its solution can be written with accuracy up to an arbitrary constant

$$\varphi(x,t)=\frac{1}{\lambda_{11}}\,S(x,t)-\frac{1}{k\lambda_{11}^2}\int\limits_0^t\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)S(x,\tau)\,d\tau. \tag{1.9}$$

Collecting terms at the zeroth power of ε and equating its average over the period to zero, we obtain the averaged equation for the function u^0:

$$-\frac{\partial}{\partial x_1}\Big[\frac{1}{\lambda_{11}}\Big(\frac{\partial u^0}{\partial x_1}+\lambda_{12}\,\frac{\partial u^0}{\partial x_2}\Big)-\frac{1}{k\lambda_{11}^2}\int\limits_0^t\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)S(x,\tau)\,d\tau\Big]$$

$$-\frac{\partial}{\partial x_2}\Big[\mu\,\frac{\partial u^0}{\partial x_2}+\frac{\lambda_{12}}{\lambda_{11}}\,S(x,t)-\frac{\lambda_{12}}{k\lambda_{11}^2}\int\limits_0^t\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)S(x,\tau)\,d\tau\Big]+\frac{\partial^2u^0}{\partial t^2}=f. \tag{1.10}$$

Substituting the explicit representation of the function $S(x,t)$ into (1.10), we arrive at the integrodifferential equation for the function $u^0(x,t)$ in divergent form

$$-\frac{\partial\sigma_{11}}{\partial x_1}-\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial^2u^0}{\partial t^2}=f, \tag{1.11}$$

where

$$\sigma_{11}=\frac{1}{\lambda_{11}}\Big(\frac{\partial u^0}{\partial x_1}+\lambda_{12}\,\frac{\partial u^0}{\partial x_2}\Big)-\frac{1}{k\lambda_{11}^2}\int\limits_0^t\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)\Big(\frac{\partial u^0}{\partial x_1}+\lambda_{12}\,\frac{\partial u^0}{\partial x_2}\Big)\,d\tau,$$

$$\sigma_{22}=\mu\,\frac{\partial u^0}{\partial x_2}+\frac{\lambda_{12}}{\lambda_{11}}\Big(\frac{\partial u^0}{\partial x_1}+\lambda_{12}\,\frac{\partial u^0}{\partial x_2}\Big)-\frac{\lambda_{12}}{k\lambda_{11}^2}\int\limits_0^t\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)\Big(\frac{\partial u^0}{\partial x_1}+\lambda_{12}\,\frac{\partial u^0}{\partial x_2}\Big)\,d\tau.$$

The last relations should be considered the determining relations for the averaged medium. Supplementing Eq. (1.11) by the initial and boundary conditions

$$u(x,0)=0,\qquad \frac{\partial u}{\partial t}(x,0)=0,\qquad u(x,t)\Big|_{\partial\Omega}=0, \tag{1.12}$$

we have an averaged boundary-value problem. We set

$$b_{11}=\frac{1}{\lambda_{11}},\qquad b_{12}=\frac{\lambda_{12}}{\lambda_{11}},\qquad b_{22}=\mu+\frac{\lambda_{12}^2}{\lambda_{11}},\qquad g_k(t-\tau)=\frac{1}{k\lambda_{11}}\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big).$$

Let us now consider the behavior of the solution of problem (1.11), (1.12) for a small k. It is worth noting that in the interval $(0,t)$, the function

$$g_k(t-\tau)=\frac{1}{k\lambda_{11}}\exp\Big(-\frac{t-\tau}{k\lambda_{11}}\Big)$$

is an approximation of the δ-function for a small k, since $g_k(t-\tau)\to\delta(t-\tau)$ as $k\to+0$, according to the theory of generalized functions. From the viewpoint of the singular perturbation theory, this implies that at the initial time, the solution of the initial boundary-value problem contains an exponentially decreasing boundary layer. As $k\to+0$, the quantity σ_{11} tends to zero and σ_{22} to $\mu\,\partial u/\partial x_2$. In this case, the degenerate equation becomes

$$-\mu\,\frac{\partial^2u}{\partial x_2^2}+\frac{\partial^2u}{\partial t^2}=f. \tag{1.13}$$

Needless to say, the initial conditions for Eq. (1.13) remain the same. Thus, in the limiting case, the effect of viscosity vanishes and the spatial dimension of the problem reduces. Equation (1.13) can be considered a degenerate hyperbolic equation. This gives rise to singularities in the solution of the initial boundary-value problem for Eq. (1.13). In the domain Ω, this solution generally does not have an integrable squared derivative $\partial u/\partial x_1$.

2. Two-Dimensional Elasticity Problem. For the elasticity equations, we formulate the following initial boundary-value problem in the domain $Q = (0,T) \times \Omega$:

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(x,t)\Big|_{\partial\Omega} = 0. \tag{2.1}$$

Here Ω is a plane domain with a piecewise smooth boundary $\partial\Omega$. It is assumed that the mass forces are nonzero. The solution of problem (2.1) will be denoted by $u^\varepsilon(x,t) = (u_1^\varepsilon, u_2^\varepsilon)$. To simplify the calculations, we assume that the elastic medium is orthotropic and use the generalized Hooke's law

$$\sigma_{11} = a_{11}(\eta)e_{11} + a_{12}(\eta)e_{22}, \qquad \sigma_{22} = a_{12}(\eta)e_{11} + a_{22}(\eta)e_{22},$$

$$\sigma_{12} = 2a_{66}(\eta)e_{12}, \qquad \eta = x_1/\varepsilon, \qquad e_{ij} = (u_{i,x_j} + u_{j,x_i})/2, \quad i,j = 1,2.$$

The functions $a_{ij}(\eta)$ are considered measurable bounded functions of the variable η under the usual assumption of positive definiteness of the elastic constant matrix. Generally, the periodicity cell $(0,1)$ consists of two materials, so that

$$a_{ij}(\eta) = a_{ij}^1, \quad \eta \in (0,h), \qquad a_{ij}(\eta) = a_{ij}^2, \quad \eta \in (h,1), \qquad i,j = 1,2,6.$$

At the interlayer boundary, the matching conditions with dissipation are imposed:

$$\sigma_{12}^\varepsilon\Big|_{\eta=h\pm 0} = \frac{k}{\varepsilon}\Big[\frac{u_2^\varepsilon}{\partial t}\Big], \qquad [\sigma_{11}^\varepsilon]\Big|_{\eta=h\pm 0} = 0, \qquad [u_1^\varepsilon] = 0, \qquad k > 0, \quad 0 < h < 1.$$

As usual, for a small ε, the solution of the problem is written as a series:

$$u_k^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_k^n(x,y,\eta), \qquad k = 1,2.$$

The equations of motion have the form

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = \rho\Big(\frac{x_1}{\varepsilon}\Big)\frac{\partial^2 u_1}{\partial t^2} + f_1(x,t), \qquad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = \rho\Big(\frac{x_1}{\varepsilon}\Big)\frac{\partial^2 u_2}{\partial t^2} + f_2(x,t), \tag{2.2}$$

where $\rho(x_1/\varepsilon)$ is the density of the material. As in Sec. 1, we replace the derivative with respect to x_1 by the total derivative $\partial/\partial x_1 + (1/\varepsilon)\,\partial/\partial\eta$ in system (2.4) and collect terms of equal powers of ε. For a power of ε^{-2}, we have the relations

$$\frac{\partial}{\partial\eta}\Big[a_{11}(\eta)\,\frac{\partial u_1^0}{\partial\eta}\Big] = 0, \qquad \frac{\partial}{\partial\eta}\Big[2a_{66}(\eta)\,\frac{\partial u_1^0}{\partial\eta}\Big] = 0,$$

which imply that $u_k^0(x,t,\eta) = u_k^0(x,t)$ $(k = 1,2)$. To determine u_1^1, we obtain the equation

$$\frac{\partial}{\partial\eta}\Big[a_{11}(\eta)\,\frac{\partial u_1^1}{\partial\eta} + a_{11}(\eta)\,\frac{\partial u_1^0}{\partial x_1} + a_{12}(\eta)\,\frac{\partial u_2^0}{\partial x_2}\Big] + \frac{\partial}{\partial x_1}\Big[a_{11}(\eta)\,\frac{\partial u_1^0}{\partial\eta}\Big] + \frac{\partial}{\partial x_2}\Big[2a_{66}(\eta)\,\frac{\partial u_2^0}{\partial\eta}\Big] = 0.$$

The function $u_1^1(x,t,\eta)$ is uniquely determined from the periodicity condition $u_1^1(x,t,0) = u_1^1(x,t,1)$, the continuity condition on the interface

$$u_1^1(x,t,h+0) = u_1^1(x,t,h-0),$$

and the continuity condition for the stress σ_{11} for $h = 0$. The function $u_2^1(x,t,\eta)$ is determined from the equation

$$\frac{\partial}{\partial\eta}\Big[2a_{66}(\eta)\,\frac{\partial u_2^1}{\partial\eta} + 2a_{66}\,(\eta)\frac{\partial u_2^0}{\partial x_1} + 2a_{66}(\eta)\,\frac{\partial u_1^0}{\partial x_2}\Big] + \frac{\partial}{\partial x_1}\Big[2a_{66}(\eta)\,\frac{\partial u_2^0}{\partial\eta}\Big] = 0$$

and the matching conditions

$$\Big(2a_{66}(\eta)\,\frac{\partial u_2^1}{\partial\eta} + 2a_{66}(\eta)\,\frac{\partial u_2^0}{\partial x_1} + 2a_{66}(\eta)\,\frac{\partial u_1^0}{\partial x_2}\Big)\Big|_{\eta=h\pm 0} = k_2\Big[\frac{\partial u_2^1}{\partial t}\Big].$$

It is clear that the function $u_2^1(x,t,\eta)$ is discontinuous in the interval $(0,1)$. Similarly as in Sec. 1, we set

$$2a_{66}(\eta)\,\frac{\partial u_2^1}{\partial\eta} + 2a_{66}(\eta)\,\frac{\partial u_2^0}{\partial x_1} + 2a_{66}(\eta)\,\frac{\partial u_1^0}{\partial x_2} = \varphi_1(x,t), \qquad S_1(x,t) = \frac{\partial u_2^0}{\partial x_1} + \frac{\partial u_1^0}{\partial x_2}.$$

After some manipulations, we obtain

$$\varphi_1(x,t) = \frac{1}{\lambda_{66}}\,S_1(x,t) - \frac{1}{k\lambda_{66}^2}\int_0^t \exp\Big(-\frac{t-\tau}{k\lambda_{66}}\Big)S_1(x,\tau)\,d\tau.$$

Collecting terms at the zeroth power of ε, we obtain averaged relations. The averaged stresses will be denoted by τ_{ij}^k $(i, j = 1, 2)$. The equations of motion have the form

$$\frac{\partial \tau_{11}^k}{\partial x_1} + \frac{\partial \varphi_1(x,t)}{\partial x_2} = \bar{\rho}\,\frac{\partial^2 u_1^k}{\partial t^2} + f_1(x,t), \qquad \frac{\partial \varphi_1(x,t)}{\partial x_1} + \frac{\partial \tau_{22}^k}{\partial x_2} = \bar{\rho}\,\frac{\partial^2 u_2^k}{\partial t^2} + f_2(x,t).$$

Here $\bar{\rho}$ is the period average of the material density and u_i^k $(i = 1, 2)$ are the displacements in the averaged problem. The stresses in the averaged problem are given by

$$\tau_{11}^k(x,t) = \frac{1}{\lambda_{11}}\,\frac{\partial u_1^k}{\partial x_1} + \frac{\lambda_{12}}{\lambda_{11}}\,\frac{\partial u_2^k}{\partial x_2}; \tag{2.3}$$

$$\tau_{12}^k(x,t) = \frac{1}{\lambda_{66}}\,S_1(x,t) - \frac{1}{k\lambda_{66}^2}\int\limits_0^t \exp\Big(-\frac{t-\tau}{k\lambda_{66}}\Big) S_1(x,\tau)\,d\tau; \tag{2.4}$$

$$\tau_{22}^k(x,t) = \frac{\lambda_{12}}{\lambda_{11}}\,\frac{\partial u_1^k}{\partial x_1} + \Big(\mu + \frac{\lambda_{12}^2}{\lambda_{11}}\Big)\frac{\partial u_2^k}{\partial x_2}. \tag{2.5}$$

Here

$$\lambda_{11} = \int\limits_0^1 \frac{1}{a_{11}(s)}\,ds, \qquad \lambda_{12} = \int\limits_0^1 \frac{a_{12}(s)}{a_{11}(s)}\,ds, \qquad \mu = \int\limits_0^1 \Big(a_{22}(s) - \frac{a_{12}(s)^2}{a_{11}(s)}\Big)\,ds, \qquad \lambda_{66} = \frac{1}{2}\int\limits_0^1 \frac{1}{a_{66}(s)}\,ds.$$

The structure of formulas (2.3)–(2.5) is clear: the stresses in the averaged problem are calculated by summing the averaged stresses corresponding to the stationary problem and viscoelastic terms. It is obvious that in the stationary problem, the elastic constant matrix is positive definite. We note that the viscoelastic term only appears in formula (2.4) for the shear stress.

Let us consider the behavior of the solution of the initial boundary-value problem for the system of equations of motion for a small k. As $k \to +0$, the shear stress τ_{12}^k tends to zero, the viscoelastic term vanishes, and the system of equations of motion becomes

$$\begin{gathered} \frac{\partial}{\partial x_1}\Big(\frac{1}{\lambda_{11}}\frac{\partial v_1}{\partial x_1} + \frac{\lambda_{12}}{\lambda_{11}}\frac{\partial v_2}{\partial x_2}\Big) = \bar{\rho}\,\frac{\partial^2 v_1}{\partial t^2} + f_1(x,t), \\ \frac{\partial}{\partial x_2}\Big(\frac{\lambda_{12}}{\lambda_{11}}\,\frac{\partial v_1}{\partial x_1} + \Big(\mu + \frac{\lambda_{12}^2}{\lambda_{11}}\Big)\frac{\partial v_2}{\partial x_2}\Big) = \bar{\rho}\,\frac{\partial^2 v_2}{\partial t^2} + f_2(x,t) \end{gathered} \tag{2.6}$$

(v_1 and v_2 are the displacement vectors for $k = 0$). Formula (2.6) does not contain the shear stress τ_{12}, the derivative of v_1 with respect to x_2 and the derivative of v_2 with respect to x_1. As in Sec. 1, this system of equations is degenerate. As shown in [1], in the stationary case, this system is a hyperbolic system with two double families of characteristics $x_1 = \text{const}$ and $x_2 = \text{const}$, its characteristic form being nonnegative.

Conclusions. A simplified case of a layered medium (compared to the general periodic problem) was considered. It was shown that the presence of dissipation in dynamic problems changes the equation of state. For low dissipation, the effect of viscoelasticity vanishes in the limiting case but the limiting equations are degenerate for the spatial variables. Clearly, this is also the case in the general periodic averaging problem and in the problem for a cell, one can obtain a singular perturbation in the time coordinate since the shear stress in this case satisfies relations of the type of (1.8).

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